Session 1 - Summary of definitions

1. Formal Preliminaries: Formal Language, Truth at a World, and Logical Consequence

Definition 1 (Language $\mathcal{L}$).

A formal language $\mathcal{L}$ is defined as follows. Let Φ be a (countable) set of objects called "propositional variables", that is $\Phi := \{p_1, p_2, p_3, \ldots\}$. Then, we define $\mathcal{L}$ via induction as follows.

1. Base: $\Phi \subseteq \mathcal{L}$.
2. Step: For all $\phi, \psi \in \mathcal{L}$:
    1. If $\phi \in \mathcal{L}$, then $\neg\phi \in \mathcal{L}$;
    2. If $\phi \in \mathcal{L}$ and $\psi \in \mathcal{L}$ then $\phi \wedge \psi \in \mathcal{L}$;
    3. If $\phi \in \mathcal{L}$ and $\psi \in \mathcal{L}$ then $\phi \vee \psi \in \mathcal{L}$;
    4. If $\phi \in \mathcal{L}$ and $\psi \in \mathcal{L}$ then $\phi \to \psi \in \mathcal{L}$.
3. Nothing else is in $\mathcal{L}$.

Definition 2 (Interpretations and Truth ($\models$)).

Let $\mathcal{L}$ be given. A possible world (or interpretation) w is a function $w : \Phi \to \{0, 1\}$ that assigns a truth value to each propositional variable $p \in \Phi$. Let W be the set of **all** possible worlds.

We define the satisfaction relation $w \models \phi$ (read as "ϕ is true at w") via induction on the structure of ϕ:

1. Base: For all $p \in \Phi$, $w \models p \iff w(p) = 1$.
2. Step: For all $\phi, \psi \in \mathcal{L}$:
    1. $w \models \neg\phi \iff w \not\models \phi$;
    2. $w \models \phi \wedge \psi \iff w \models \phi \text{ and } w \models \psi$.

2. Properties of Logical Consequence Relations and Operators

2.1. Logical Consequence

Definition 3 (Logical Consequence ($\vdash$)).

Let $\phi, \psi \in \mathcal{L}$ and W be given. We say that ψ follows logically from ϕ, $\phi \vdash \psi$, iff the latter is true at all the worlds w that make the former true. In symbols:

$$\phi \vdash \psi \quad :\iff \quad \text{For all } w \in W : w \models \phi \implies w \models \psi$$

Definition 4 (Consequence Operator Cn).

Let $\vdash$ be given. We define the consequence operator Cn (of $\vdash$) as a function from sets of sentences $A \subseteq \mathcal{L}$ to sets of sentences $B \subseteq \mathcal{L}$:

$$Cn(A) := \{\phi \in \mathcal{L} : A \vdash \phi\}$$

N.b., Cn is a function from $\mathcal{P}(\mathcal{L})$ to $\mathcal{P}(\mathcal{L})$.

Definition 5 (Belief Sets).

Let B be a subset of $\mathcal{L}$. B is a belief set if, and only if, the following equivalence holds:

$$B = Cn(B)$$

Let us call $\mathbf{B}$ the set of all belief sets.

2.2. Properties of Logical Consequence Relation and Operator

Proposition 6.

Let $\vdash$ be the consequence relation defined in [Definition 3 (Logical Consequence ($\vdash$))]. $\vdash$ is

1. *Reflexive:* For all $\gamma \in \Gamma : \Gamma \triangleright \gamma$
2. *Transitive:* If $\Gamma \triangleright \delta$ for all $\delta \in \Delta$, and $\Delta \triangleright \phi \implies \Gamma \triangleright \phi$.
3. *Monotonic:* If $\Gamma \triangleright \phi$ and $\Gamma \subseteq \Delta \implies \Delta \triangleright \phi$.

Proposition 7.

Let $\vdash$ be the consequence relation defined in [Definition 3 (Logical Consequence ($\vdash$))], and Cn the consequence operator defined in [Definition 4 (Consequence Operator Cn)] based on $\vdash$. Cn satisfies the following properties:

1. ***Reflexivity**: For all $\Gamma \subseteq \mathcal{L}$, $\Gamma \subseteq Cn(\Gamma)$.*
2. ***Transitivity**: For all $\Gamma, \Delta \subseteq \mathcal{L}$, if $\Delta \subseteq Cn(\Gamma)$ then $Cn(\Delta) \subseteq Cn(\Gamma)$.*
3. ***Monotonicity**: For all $\Gamma, \Delta \subseteq \mathcal{L}$, if $\Delta \subseteq \Gamma$ then $Cn(\Delta) \subseteq Cn(\Gamma)$.*
4. ***Idempotence**: For all $\Gamma \subseteq \mathcal{L}$, $Cn(Cn(\Gamma)) = Cn(\Gamma)$.*

Proposition 8.

Let $\vdash$ be the consequence relation defined in [Definition 3 (Logical Consequence ($\vdash$))]. The following properties hold for $\vdash$.

1. ***Compactness**: $\Gamma \vdash \phi \implies \exists \Gamma_{fin} \subseteq \Gamma$ such that $\Gamma_{fin} \vdash \phi$, where Γ_{fin} is a finite subset of Γ.*
2. ***Deduction Theorem**: If $\Gamma \cup \{\phi\} \vdash \psi$, then $\Gamma \vdash \phi \to \psi$.*
3. ***Disjunction in the Premises**: If $\Gamma \cup \{\phi_1\} \vdash \psi$ and $\Gamma \cup \{\phi_2\} \vdash \psi$, then $\Gamma \cup \{\phi_1 \vee \phi_2\} \vdash \psi$.*

3. Belief Revision Theory

Let $B \in \mathbf{B}$ be any belief set, $\phi \in \mathcal{L}$ be any sentence, and $* : \mathbf{B} \times \mathcal{L} \to \mathbf{B}$ a candidate revision function. $*$ is a (AGM) **(basic) belief revision function** iff it satisfies the following postulates:

$$B * \phi \text{ is a belief set} \qquad \text{(Closure)}$$

$$\phi \in B * \phi \qquad \text{(Success)}$$

$$\text{If } \phi \nvdash \bot, \text{ then } Cn(B * \phi) \neq \mathcal{L} \qquad \text{(Consistency)}$$

$$B * \phi \subseteq Cn(B \cup \{\phi\}) \qquad \text{(Inclusion)}$$

$$\text{If } B \nvdash \neg\phi, \text{ then } B * \phi = Cn(B \cup \phi) \qquad \text{(Vacuity)}$$

$$\text{If } Cn(\{\phi\}) = Cn(\{\psi\}), \text{ then } B * \phi = B * \psi \qquad \text{(Congruence)}$$

$*$ is a **belief revision function** iff it satisfies also the following postulates

$$B * (\phi \wedge \psi) \subseteq (B * \phi) + \psi \qquad \text{(Superexpansion)}$$

$$\text{If } B * \phi \nvdash \neg\psi, \text{ then } (B * \phi) + \psi \subseteq B * (\phi \wedge \psi) \qquad \text{(Subexpansion)}$$